Exact ordinary differential equations

We say that an ODE is exact if its general representation is:

$$P(x,y)\cdot dx + Q(x,y)\cdot dy = 0$$

And there must exist a function $U(x,y)$ called the potential function, such that:

$$dU(x,y) = U'_x dx + U'_y dy \quad \Rightarrow \quad U'_x = P(x,y) \quad \text{and} \quad U'_y = Q(x,y)$$

Since $dU(x,y) = 0$ when the functions $P(x,y)$ and $Q(x,y)$ are equal to the partial derivatives of $U(x,y)$:

$$U(x,y) = C$$

For the equation to be exact, it must satisfy the symmetry condition: $P'_y = Q'_x$

$$\begin{cases} \frac{\partial U}{\partial x} = P(x,y) \rightarrow \int dU = \int P(x,y)\,dx = F(x,y) + \alpha(y) \\ \frac{\partial U}{\partial y} = Q(x,y) \rightarrow \int dU = \int Q(x,y)\,dy = F(x,y) + \beta(x) \end{cases}$$

$$\Rightarrow U(x,y) = F(x,y) + \alpha(y) + \beta(x) = C$$

The integration constant depends on x, since having previously derived with respect to y, a term dependent on the other variable could have been lost.

Example:

$$(2x^3 + y)\cdot dx + (x + 2y^2)\cdot dy = 0$$

For the equation to be exact, it must satisfy the symmetry condition: $P'_y = Q'_x$

$$P'_y = 1 \quad \text{and} \quad Q'_x = 1 \quad \Rightarrow \quad 1 = 1$$

$$\begin{cases} \frac{\partial U}{\partial x} = (2x^3 + y) \rightarrow \int dU = \int (2x^3 + y)\,dx = 2\frac{x^4}{4} + xy = \frac{x^4}{2} + xy \\ \frac{\partial U}{\partial y} = (x + 2y^2) \rightarrow \int dU = \int (x + 2y^2)\,dy = xy + 2\frac{y^3}{3} \end{cases}$$

$$\Rightarrow U(x,y) = F(x,y) + \alpha(y) + \beta(x) = xy + \frac{x^4}{2} + 2\frac{y^3}{3} = C$$

Solution:

$$xy + \frac{x^4}{2} + 2\frac{y^3}{3} = C$$

Example 2:

$$(3y + e^x)dx + (3x + \cos(y))dy = 0$$

Let's verify if it is exact.

$$P'_y = 3 \quad \text{and} \quad Q'_x = 3 \quad \Rightarrow \quad P'_y = Q'_x$$

We calculate:

$$U(x,y) = \int P(x,y)dx = \int (3y + e^x)dx = 3yx + e^x$$

$$U(x,y) = \int Q(x,y)dy = \int (3x + \cos(y))dy = 3xy + \sin y$$

Finally, the solution to the differential equation is:

$$3yx + e^x + \sin(y) = C$$